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On a class of dimensionless groups in dimensional analysis

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Dedicated to the 85th birthday of Professor Géza Schay, member of the Hungarian Academy of Sciences

1. INTRODUCTION AND SUMMARY

THE CONCEPT of a physical quantity, say A, and its dimension is taken to be known (see, e.g. [1], and also regarding the historical background [2]). Thus, given A and the *fundamental quantities* $B_1, B_2, \ldots, B_m$ (e.g. mass, length, time, etc.), the dimension of A, denoted by $[A]$, is a product of powers of the B_i $(i = 1, 2, \ldots, m)$:

$$[A] = B_1^{\alpha_1} B_2^{\alpha_2} \ldots B_m^{\alpha_m} = \prod_{i=1}^{m} B_i^{\alpha_i}. \tag{1}$$

Consequently, for a set of physical quantities $A_1, A_2, \ldots, A_n$ we have

$$[A_j] = \prod_{i=1}^{m} B_i^{a_{ij}}; \quad j = 1, 2, \ldots, n. \tag{2}$$

A *dimensionless group* will then be a product of powers of A_j:

$$X = \prod_{j=1}^{n} A_j^{x_j} \tag{3}$$

where the x_j should be such that they solve the system of linear equations

$$\sum_{j=1}^{n} a_{ij} x_j = 0; \quad i = 1, 2, \ldots, m. \tag{4}$$

Since

$$[X] = \prod_{j=1}^{n} [A_j]^{x_j} = \prod_{j=1}^{n} \prod_{i=1}^{m} B_i^{a_{ij}x_j}$$

$$= \prod_{i=1}^{m} \prod_{j=1}^{n} B_i^{a_{ij}x_j} = \prod_{i=1}^{m} B_i^{\sum_{j=1}^{n} a_{ij}x_j} = 1$$

it is evident that X is 'dimensionless'. (In what follows we simply say that X is a group.)

An example. Let us consider in a fluid dynamical problem the following physical quantities:

A_1 : tube diameter
A_2 : linear velocity
A_3 : fluid density
A_4 : fluid viscosity

as well as the fundamental quantities

B_1 : mass
B_2 : length
B_3 : time.

Now we have for (2)

$$[A_1] = B_2, \quad [A_2] = B_2 B_3^{-1}, \quad [A_3] = B_1 B_2^{-3},$$
$$[A_4] = B_1 B_2^{-1} B_3^{-1} \tag{5}$$

and for (3), for example,

$$X = A_1 A_2 A_3 A_4^{-1} \quad \text{('Reynolds number')} \tag{6}$$

where in fact, $[X] = 1$. Actually, (6) will be the *only* dimensionless group in the problem if groups corresponding to different powers of X are considered identical.

Generally, however, the number of all (possible) groups in a given problem will be infinite. But it is enough to consider only linearly independent groups (see Section 3) of maximum number, a so-called base, since each group can then be represented as a unique linear combination of the particular groups in the respective base. However, the number of all bases becomes, as a rule, infinite. In the following, so-called *simple* groups will be defined, which are a finite collection of groups, the maximum number of linearly independent groups therein being unchanged. These 'simple' groups appear in a quite natural way in the search for all possible groups in a particular problem. A straightforward computer program to find all the simple groups in a set of physical quantities has been developed.

Dimensional analysis as treated in this paper, is 'isomorphic' to the formal stoicheiometry of chemical species and their reactions, as far as the linear algebraic aspect is concerned [3, 4]. The basic idea goes back to the early 1960s [5, 6].

2. THE MATHEMATICAL PROBLEM

2.1. The general solution of the system of homogeneous linear equations (4) is to be found. By introducing the m-dimensional column vectors

$$\mathbf{a}_1 = [a_{i1}], \ldots, \mathbf{a}_n = [a_{in}]; \quad i = 1, 2, \ldots, m \tag{7}$$

equation (4) can be written in a more compact form:

$$\sum_{j=1}^{n} \mathbf{a}_j x_j = 0. \tag{8}$$

As is well known, the general solution of equation (4) can be expressed as

$$\mathbf{x} = \sum_{k=1}^{n-r} \lambda_k \mathbf{x}_k \tag{9}$$

where r is the rank of the matrix of coefficients $[a_{ij}]$, $\mathbf{x}$ is the solution vector:

$$\mathbf{x} = [x_1, x_2, \ldots, x_n]^T \tag{10}$$

and, further, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-r}$ are linearly independent particular solutions and $\lambda_1, \lambda_2, \ldots, \lambda_{n-r}$ are arbitrary parameters (free variables).

2.2. The representation (9) is evidently only one of the infinite number of possibilities. Next, a subset of all particular solutions, i.e. that of the so-called *basic* solutions, is defined. This happens in the following way. The system of equations (8) is solved according to Cramer's rule, i.e. a *base*

$$\{\mathbf{a}_{j_1}, \mathbf{a}_{j_2}, \ldots, \mathbf{a}_{j_r}\}; \quad r = \text{rank}\,[a_{ij}] \tag{11}$$

is chosen and the following system of equations is considered:

$$\mathbf{a}_{j_1} x_{j_1} + \ldots + \mathbf{a}_{j_r} x_{j_r} = \mathbf{a}_{j_k}; \quad k = r+1, \ldots, n \tag{12}$$
$$x_{j_l} = 0; \quad l \neq 1, 2, \ldots, r, k \quad (k \text{ being fixed}).$$

Obviously, the solution of the nonhomogeneous system (12), also called a *basic* solution, is unique. Moreover, by choosing different $\mathbf{a}_{j_k}$ $(k = r+1, \ldots, n)$ on the RHS of equation (12), a total of $n-r$ linearly independent basic solutions are obtained [6]. The general solution of equation (4) can then be written in the form of equation (9).

2.3. So far the base (11) has not been varied. What will happen if different bases are chosen and the corresponding basic solutions are sought? Evidently, the set of all basic solutions is thus obtained, which are no more linearly independent, however, the maximum number of linearly independent basic solutions remains $n-r$.

Definition. A family of vectors is called a simplex if the vectors possess the following properties: (a) they are linearly dependent, and (b) omitting any one of them the rest forms a linearly independent system.

Theorem [6]. Let us consider a simplex among the vectors (7), e.g.

$$\{\mathbf{a}_{j_1}, \mathbf{a}_{j_2}, \ldots, \mathbf{a}_{j_p}\} \quad (p \leqslant n) \tag{13}$$

and the respective system of equations

$$\mathbf{a}_{j_1}x_{j_1} + \ldots + \mathbf{a}_{j_p}x_{j_p} = 0 \tag{14}$$
$$x_{j_l} = 0 (l \neq 1, 2, \ldots, p).$$

Let us call the solution of this system *simple*. If we now consider all the simple solutions, they turn out to be identical with the basic solutions defined in 2.2.

3. SIMPLE GROUPS IN DIMENSIONAL ANALYSIS

In order to make possible a mathematical treatment of the search for all (dimensionless) groups in a physical problem, the vectors (7) are introduced and $\mathbf{a}_j$ is identified with the physical quantity A_j; furthermore a group X among the physical quantities considered is identified with a solution $\mathbf{x}$ of equation (4). Naturally, (a) the trivial group ($\mathbf{x} = 0$) is disregarded, and (b) two linearly dependent groups are considered identical. Further, the maximum number of linearly independent groups becomes

$$n - \text{rank}\,[a_{ij}]. \tag{15}$$

As a trivial example, let us again take (5), i.e. with B_1: mass, B_2: length and B_3: time:

$$[a_{ij}] = \begin{array}{c|cccc} & A_1 & A_2 & A_3 & A_4 \\ \hline B_1 & 0 & 0 & 1 & 1 \\ B_2 & 1 & 1 & -3 & -1 \\ B_3 & 0 & -1 & 0 & -1 \end{array} \; ; \text{rank}\,[a_{ij}] = 3. \tag{16}$$

According to (15) the solution of equation (4) is unique:

$$\mathbf{x} = [1, 1, 1, -1]^{\mathrm{T}},$$

i.e. the only group (and the only simple group at the same time) becomes (6).

Let us now extend the above example by adding to the physical quantities (5) two more, thereby obtaining the following list:

A_1: tube diameter
A_2: linear velocity
A_3: fluid density
A_4: fluid viscosity
A_5: acceleration due to gravity
A_6: pressure drop in the tube.

For the system matrix in (4) we thus have:

$$[a_{ij}] = \begin{array}{c|cccccc} & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ \hline B_1 & 0 & 0 & 1 & 1 & 0 & 1 \\ B_2 & 1 & 1 & -3 & -1 & 1 & -1 \\ B_3 & 0 & -1 & 0 & -1 & -2 & -2 \end{array} \; ; \text{rank}\,[a_{ij}] = 3. \tag{17}$$

The set of all simple groups can be obtained, according to the theorem (see Section 2), by determining all simplexes among the column vectors of matrix (17), see Fig. 1. Exactly the following eleven simplexes have been found (for, e.g. $\{A_1, A_2, A_5\}$, we simply write $\{1, 2, 5\}$):

{1, 2, 5}	'Froude'	... 1
{2, 3, 6}	'Euler'	... 2
{1, 2, 3, 4}	'Reynolds'	... 3
{1, 2, 4, 6}	'Poiseuille'	... 4
{1, 3, 4, 5}	'Galilei'	... 5
{1, 3, 4, 6}	'Kármán'	... 6
{1, 3, 5, 6}	'Bernoulli'	... 7
{1, 4, 5, 6}	'd'Alembert'	... 8
{2, 3, 4, 5}	'Lagrange'	... 9
{2, 4, 5, 6}	'Navier'	... 10
{3, 4, 5, 6}	'Laplace'	... 11

Only the names 'Froude', 'Euler', 'Reynolds' and 'Galilei' are used in the literature (for the respective groups, of course), the others being suggestions of the authors of this paper. The corresponding simple groups are the column vectors of the following matrix:

$$[\mathbf{x}_1, \ldots, \mathbf{x}_{11}] = \begin{array}{c|ccccccccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} \\ \hline A_1 & 1 & 0 & 1 & 1 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ A_2 & -2 & 2 & 1 & -1 & 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ A_3 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & 0 & 1 & 0 & 1 \\ A_4 & 0 & 0 & -1 & -1 & -2 & -2 & 0 & -2 & -1 & -1 & 2 \\ A_5 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 & 2 \\ A_6 & 0 & -1 & 0 & 1 & 0 & 1 & -1 & 2 & 0 & 1 & -3 \end{array} \tag{18}$$

The maximum number of linearly independent groups is given by (16):

$$\text{rank}\,[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{11}] = n\text{-rank}\,[a_{ij}] = 3. \tag{19}$$

At this stage of the development one might enquire about the possible *relations* (more exactly, simple relations) between the groups considered in (18). In fact, the mathematics remains the same as was for the search of simple groups between

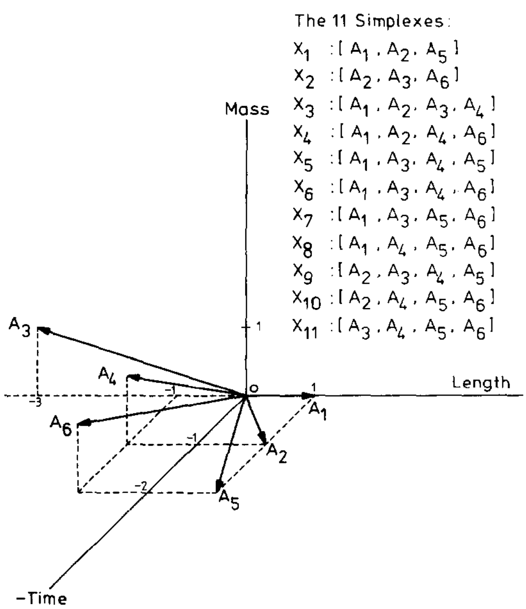

FIG. 1. Space of simplexes.

physical quantities (the term 'relation' being used instead of speaking about 'groups of groups'); e.g., one would find that

$$X_1X_3^2X_5^{-1} = 1$$

constitutes a simple relation. Once again, one could write down the set of all simple relations and continue with the hierarchy of groups, relations, and so on. However, this procedure seems to be a delicacy for the mathematically interested reader only.

Finally, a remark about the boundaries of dimensional analysis is due. Taking a *triplet* of linearly independent groups, see equation (18), e.g. $\{X_1, X_2, X_3\}$, by dimensional analysis we only know that there is a functional relationship between the groups considered, say

$$f(X_1, X_2, X_3) = 0.$$

Only further investigations (experimental or theoretical) can give more information about the actual form of this relationship.

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APPENDIX: EXAMPLE FOR A GROUP THAT IS NOT SIMPLE

A nonsimple group might be characterised in such a way that it can always be 'simplified', i.e. roughly speaking, by omitting some of the physical quantities occurring in it, the rest will constitute a simple group; e.g., in the former example, a nonsimple group is the following:

$$X = A_1^2A_2^{-3}A_4^{-1}A_5A_6.$$

Check: $\mathbf{x} = [2, -3, 0, -1, 1, 1,]^T$ is in fact a particular solution of equation (4) with (17).

X can be simplified in several ways:

(1) Deleting A_1, the rest constitutes 'Navier'.
(2) Deleting A_2, the rest constitutes 'd'Alembert'.
(3) Deleting A_5, the rest constitutes 'Poiseuille'.
(4) Deleting A_4 and A_6, the rest constitutes 'Froude'.

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Temperature ratio effects in compressible turbulent boundary layers

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1. INTRODUCTION

THIS PAPER describes an experimental and theoretical investigation of the heat transfer coefficient for a compressible, constant pressure, turbulent boundary layer on an isothermal flat plate. In the presence of large temperature differences between the freestream and the wall, the Nusselt number can be expected to depend on the ratio $T_w/T_{t\infty}$, due to compressibility effects and variations in gas properties with temperature through the boundary layer. At constant freestream Reynolds number, this is generally written in the form

$$Nu = Nu_i(T_w/T_{t\infty})^n.$$

There is a lack of experimental data in the literature, but reported analytical work for air tends to suggest a decrease in Nu with wall-to-gas temperature ratio. Kays and Crawford [1], for example, give a value for n of -0.4 for $T_w/T_{t\infty} > 1$; Eckert's reference temperature [2] corresponds to an exponent of -0.19 for the conditions investigated in this paper. Brown's computations of flat plate heat transfer [3] also show a decrease of Nu with $T_w/T_{t\infty}$, as do the turbulent heat transfer charts presented by Neal and Bertram [4]. Bose [5] solves the turbulent boundary-layer equations numerically for $0.1 < T_w/T_{t\infty} < 0.9$ and lists different St–Re correlations for the three temperature ratios which he considers. In all the cases described above, a negative value for the exponent n could be inferred. Previous experimental work at Oxford by Loftus and Jones [6] suggested that this effect was small.

This paper examines the mechanisms for the dependence of Nusselt number on wall-to-gas temperature ratio. In addition, experimental results are presented for air at $M = 0.55$ and $Re/m = 2.7 \times 10^7\ \mathrm{m}^{-1}$, for $0.5 < T_w/T_{t\infty} < 1.3$. This data is compared with numerical solutions of the turbulent, compressible boundary-layer equations using conventional mixing length turbulence models.

2. ANALYTICAL DISCUSSION OF THE TEMPERATURE RATIO EFFECT

Although the effect of the wall-to-gas temperature ratio is complicated, some understanding of possible mechanisms for producing such changes in the Nusselt number can be gained from a study of the laminar compressible boundary-layer equations, which for convenience can be considered in the simplified form

$$\left.\begin{aligned} & ff'' + (Cf'')' = 0 \\ & [(C/Pr)T']' + fT' + (\gamma-1)M_\infty^2 C T_\infty f''^2 = 0 \end{aligned}\right\} \quad (1)$$

with boundary conditions

$$f(0) = f'(0) = 0, \quad T(0) = T_w, \quad f'(\infty) = 1, \quad T(\infty) = T_\infty$$

where

$$' = \frac{d}{ds}, \quad T = T(s), \quad \Psi = \sqrt{(2\mu_\infty U_\infty x/\rho_\infty)}f(s),$$
$$C = \mu\rho/\mu_\infty\rho_\infty$$